

Uniform structures on differential spaces

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January 14, 2013

Abstract

The uniform structure on a differential space defined by a family of generators is considered.

Key words and phrases: differential space, uniform structure.

2000 AMS Subject Classification Code 58A40.

1 Introduction

This paper is the second of the series of publications concerning integration of differential forms and densities on differential spaces (the first one is [6]). It has the preliminary character. We recall basic facts from the theory of differential spaces and the theory of uniform structures. After that we describe uniform structures defined on a differential space by families of generators of its differential structure.

Section 2 of the paper contains basic definitions and the description of preliminary facts concerning theory of differential spaces. Foundations of theory of differential spaces can be find in [5]. In Section 3 we give basic definitions and describe the standard facts concerning theory of uniform spaces. We define (in a standard manner) the uniform structure given on a differential space by a family of generators of its differential structure. Section 4 contains basic facts concerning uniform (uniformly continuous) maps. In Section 5 we recall the definition of a complete uniform space and the standard construction of completion of a given uniform space. Here we introduce and investigate the notion of the extension of a differential structure.

Without any other explanation we use the following symbols: \mathbf{N} -the set of natural numbers; \mathbf{R} -the set of reals.

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2 Differential spaces

Let M be a nonempty set and let \mathcal{C} be a family of real valued functions on M . Denote by $\tau_{\mathcal{C}}$ the weakest topology on M with respect to which all functions of \mathcal{C} are continuous.

A base of the topology $\tau_{\mathcal{C}}$ consists of sets:

$$(\alpha_1, \dots, \alpha_n)^{-1}(P) = \bigcap_{i=1}^n \{p : a_i < \alpha_i(p) < b_i\},$$

where $n \in \mathbf{N}$, $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{R}$, $a_i < b_i$, $\alpha_1, \dots, \alpha_n \in \mathcal{C}$, $P = \{(x_1, \dots, x_n) \in \mathbf{R}^n; a_i < x_i < b_i, i = 1, \dots, n\}$.

DEFINITION 2.1 A function $f : M \rightarrow \mathbf{R}$ is called a *local \mathcal{C} -function on M* if for every $p \in M$ there is a neighbourhood V of p and $\alpha \in \mathcal{C}$ such that $f|_V = \alpha|_V$. The set of all local \mathcal{C} -functions on M is denoted by \mathcal{C}_M .

Note that any function $f \in \mathcal{C}_M$ is continuous with respect to the topology $\tau_{\mathcal{C}}$. In fact, if $\{V_i\}_{i \in I}$ is such an open (with respect to $\tau_{\mathcal{C}}$) covering of M that for any $i \in I$ there exists $\alpha_i \in \mathcal{C}$ satisfying $f|_{V_i} = \alpha_i|_{V_i}$ and U is an open subset of \mathbf{R} then

$$f^{-1}(U) = \bigcup_{i \in I} (\alpha_i|_{V_i})^{-1}(U).$$

Since $(\alpha_i|_{V_i})^{-1}(U)$ is open in V_i and $V_i \in \tau_{\mathcal{C}}$ we obtain $(\alpha_i|_{V_i})^{-1}(U) \in \tau_{\mathcal{C}}$ for any $i \in I$. Hence $f^{-1}(U) \in \tau_{\mathcal{C}}$. Bearing in mind that U is an arbitrary open set in \mathbf{R} we obtain that f is continuous with respect to $\tau_{\mathcal{C}}$.

We have $\mathcal{C} \subset \mathcal{C}_M$ which implies $\tau_{\mathcal{C}} \subset \tau_{\mathcal{C}_M}$. On the other hand any element of \mathcal{C}_M is a function continuous with respect to $\tau_{\mathcal{C}}$. Then $\tau_{\mathcal{C}_M} \subset \tau_{\mathcal{C}}$ and consequently $\tau_{\mathcal{C}_M} = \tau_{\mathcal{C}}$.

DEFINITION 2.2 A function $f : M \rightarrow \mathbf{R}$ is called *\mathcal{C} -smooth function on M* if there exist $n \in \mathbf{N}$, $\omega \in C^\infty(\mathbf{R}^n)$ and $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ such that

$$f = \omega \circ (\alpha_1, \dots, \alpha_n).$$

The set of all \mathcal{C} -smooth functions on M is denoted by $sc\mathcal{C}$.

We have $\mathcal{C} \subset sc\mathcal{C}$ which implies $\tau_{\mathcal{C}} \subset \tau_{sc\mathcal{C}}$. On the other hand any superposition $\omega \circ (\alpha_1, \dots, \alpha_n)$ is continuous with respect to $\tau_{\mathcal{C}}$ which gives $\tau_{sc\mathcal{C}} \subset \tau_{\mathcal{C}}$. Consequently $\tau_{sc\mathcal{C}} = \tau_{\mathcal{C}}$.

DEFINITION 2.3 A set \mathcal{C} of real functions on M is said to be a (*Sikorski's*) *differential structure* if: (i) \mathcal{C} is *closed with respect to localization* i.e. $\mathcal{C} = \mathcal{C}_M$; (ii) \mathcal{C} is *closed with respect to superposition with smooth functions* i.e. $\mathcal{C} = sc\mathcal{C}$.

In this case a pair (M, \mathcal{C}) is said to be a (*Sikorski's*) *differential space*.

PROPOSITION 2.1. *The intersection of any family of differential structures defined on a set $M \neq \emptyset$ is a differential structure on M .*

Proof. Let $\{\mathcal{C}_i\}_{i \in I}$ be a family of differential structures defined on a set M and let $\mathcal{C} := \bigcap_{i \in I} \mathcal{C}_i$. Then \mathcal{C} is nonempty family of real-valued functions on M (it contains all constant functions). If $n \in \mathbf{N}$, $\omega \in C^\infty(\mathbf{R}^n)$ and $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ then for any $i \in I$ $\alpha_1, \dots, \alpha_n \in \mathcal{C}_i$ and consequently $\omega \circ (\alpha_1, \dots, \alpha_n) \in \mathcal{C}_i$. Hence $\omega \circ (\alpha_1, \dots, \alpha_n) \in \mathcal{C}$ which means that $sc\mathcal{C} = \mathcal{C}$.

Since $\mathcal{C} \subset \mathcal{C}_i$ for any $i \in I$ we have $\tau_{\mathcal{C}} \subset \tau_{\mathcal{C}_i}$. It means that any subset of M open with respect to $\tau_{\mathcal{C}}$ is open with respect to $\tau_{\mathcal{C}_i}$, for $i \in I$.

Let $\beta \in \mathcal{C}_M$. Choose for any $p \in M$ a set $U_p \in \tau_{\mathcal{C}}$ and a function $\alpha_p \in \mathcal{C}$ such that $p \in U_p$ and $\beta|_{U_p} = \alpha_p|_{U_p}$. Since $\alpha_p \in \mathcal{C}_i$ and $U_p \in \tau_{\mathcal{C}_i}$ we obtain $\beta \in (\mathcal{C}_i)_M = \mathcal{C}_i$, for any $i \in I$. Then $\beta \in \mathcal{C}$ and consequently $\mathcal{C}_M = \mathcal{C}$.

Equalities $\mathcal{C}_M = \mathcal{C} = sc\mathcal{C}$ means that \mathcal{C} is a differential structure on M . \square

Let \mathcal{F} be a set of real functions on M . Then, by Proposition 2.1, the intersection \mathcal{C} of all differential structures on M containing \mathcal{F} is a differential structure on M . It is the smallest differential structure on M containing \mathcal{F} . One can easily prove that $\mathcal{C} = (sc\mathcal{F})_M$ (see [4]). This structure is called *the differential structure generated by \mathcal{F}* . Functions of \mathcal{F} are called *generators* of the differential structure \mathcal{C} . We have also $\tau_{(sc\mathcal{F})_M} = \tau_{sc\mathcal{F}} = \tau_{\mathcal{F}}$ (see remarks after Definitions 2.1 and 2.2).

Let (M, \mathcal{C}) and (N, \mathcal{D}) be differential spaces. A map $F : M \rightarrow N$ is said to be *smooth* if for any $\beta \in \mathcal{D}$ the superposition $\beta \circ F \in \mathcal{C}$. We will denote the fact that f is smooth writing

$$F : (M, \mathcal{C}) \rightarrow (N, \mathcal{D}).$$

If $F : (M, \mathcal{C}) \rightarrow (N, \mathcal{D})$ is a bijection and $F^{-1} : (N, \mathcal{D}) \rightarrow (M, \mathcal{C})$ then F is called a *diffeomorphism*.

It is easy to show that if \mathcal{F} is a family of generators of the structure \mathcal{D} on the set N then $F : (M, \mathcal{C}) \rightarrow (N, \mathcal{D})$ iff for any $f \in \mathcal{F}$ the superposition $f \circ F \in \mathcal{C}$.

If A is a nonempty subset of M and \mathcal{C} is a differential structure on M then \mathcal{C}_A denotes the differential structure on A generated by the family of restrictions $\{\alpha|_A : \alpha \in \mathcal{C}\}$. The differential space (A, \mathcal{C}_A) is called a *differential subspace* of (M, \mathcal{C}) . One can easily prove the following

PROPOSITION 2.2. *Let (M, \mathcal{C}) and (N, \mathcal{D}) be differential spaces and let $F : M \rightarrow N$. Then $F : (M, \mathcal{C}) \rightarrow (N, \mathcal{D})$ iff $F : (M, \mathcal{C}) \rightarrow (F(M), F(M)_{\mathcal{D}})$.*

If the map $F : (M, \mathcal{C}) \rightarrow (F(M), F(M)_{\mathcal{D}})$ is a diffeomorphism then we say that $F : M \rightarrow N$ is a *diffeomorphism onto its range* (in (N, \mathcal{D})). In particular the natural embedding $A \ni x \mapsto i(x) := x \in M$ is a diffeomorphism of (A, \mathcal{C}_A) onto its range in (M, \mathcal{C}) .

If $\{(M_i, \mathcal{C}_i)\}_{i \in I}$ is an arbitrary family of differential spaces then we consider the Cartesian product $\prod_{i \in I} M_i$ as a differential space with the differential structure

$\bigotimes_{i \in I} \mathcal{C}_i$ generated by the family of functions $\mathcal{F} := \{\alpha_i \circ pr_i : i \in I, \alpha_i \in \mathcal{C}_i\}$, where $\prod_{i \in I} M_i \ni (x_i) \mapsto pr_j((x_i)) =: x_j \in M_j$ for any $j \in I$. The topology $\tau_{\bigotimes_{i \in I} \mathcal{C}_i}$ coincides with the standard product topology on $\prod_{i \in I} M_i$.

A *generator embedding* of the differential space (M, \mathcal{C}) into the Cartesian space is a mapping $\phi_{\mathcal{F}} : (M, \mathcal{C}) \rightarrow (\mathbf{R}^{\mathcal{F}}, C^{\infty}(\mathbf{R}^{\mathcal{F}}))$ given by the formula

$$\phi_{\mathcal{F}}(p) = (\alpha(p))_{\alpha \in \mathcal{F}}$$

(for example if $\mathcal{F} = \{\alpha_1, \alpha_2, \alpha_3\}$ then $\phi_{\mathcal{F}}(p) = (\alpha_1(p), \alpha_2(p), \alpha_3(p)) \in \mathbf{R}^3 \cong \mathbf{R}^{\mathcal{F}}$).

PROPOSITION 2.3. *Let \mathcal{F} be a family of generators of the differential structure \mathcal{C} on the set M . If \mathcal{F} separates points of M then the generator embedding defined by \mathcal{F} is a diffeomorphism onto its image. On that image we consider a differential structure of a subspace of $(\mathbf{R}^{\mathcal{F}}, C^{\infty}(\mathbf{R}^{\mathcal{F}}))$.*

Proof. Since \mathcal{F} separates points of M it follows from the definition of differential embedding $\phi_{\mathcal{F}}$ that it is an one-to-one mapping onto its image in $\mathbf{R}^{\mathcal{F}}$. Moreover for any $f \in \mathcal{F}$ we have $pr_f \circ \phi_{\mathcal{F}} = f \in \mathcal{C}$. Since the differential structure $C^{\infty}(\mathbf{R}^{\mathcal{F}})$ is generated by the family $\{pr_g\}_{g \in \mathcal{C}}$ we obtain that the map $\phi_{\mathcal{F}}$ is smooth with respect to $C^{\infty}(\mathbf{R}^{\mathcal{F}})$. It remains to show that the map $\phi_{\mathcal{F}}^{-1}$ is smooth.

For any $f \in \mathcal{F}$ we have

$$f \circ \phi_{\mathcal{F}}^{-1} = pr_{f|_{\mathcal{F}(M)}}.$$

It means that $f \circ \phi_{\mathcal{F}}^{-1} \in C^{\infty}(\mathbf{R}^{\mathcal{F}})_{\mathcal{F}(M)}$. Since the differential structure \mathcal{C} is generated by \mathcal{F} we obtain $\phi_{\mathcal{F}}^{-1}$ is smooth. \square

3 Uniform structures

Let X be a nonempty set.

DEFINITION 3.1. A set $\Delta = \{(x, x) : x \in X\}$ is said to be *the diagonal* of the product $X \times X$. A set $V \subset X \times X$ is called a *neighbourhood of the diagonal* if $\Delta \subset V$ and $V = -V$, where $-V = \{(x, y) : (y, x) \in V\}$. A family of all neighborhoods of the diagonal is denoted by \mathcal{D}_X .

DEFINITION 3.2 If for $x, y \in X$ and $V \in \mathcal{D}_X$ we have $(x, y) \in V$, then we say that x and y are *distant less than V* ($|x - y| < V$). We say that *the diameter of a set $A \subset X$ is less than V* ($\delta(A) < V$) if for all $x, y \in A$ we have $|x - y| < V$. A *ball with the center at $x_0 \in X$ and the radius V* is a set $K(x_0, V) = \{x \in X : |x_0 - x| < V\}$. The set

$$2V := \{(x, y) \in X \times X : \exists z \in X [(x, z) \in V \wedge (x, z) \in V]\}.$$

DEFINITION 3.3 A uniform structure \mathcal{U} on X is a subfamily of \mathcal{D}_X satisfying the following conditions:

- 1) $(V \in \mathcal{U} \wedge V \subset W \in \mathcal{D}_X \Rightarrow (W \in \mathcal{U}))$;
- 2) $(V_1, V_2 \in \mathcal{U}) \Rightarrow (V_1 \cap V_2 \in \mathcal{U})$;
- 3) $\forall V \in \mathcal{U} \exists W \in \mathcal{U} [2W \subset V]$;
- 4) $\bigcap \mathcal{U} = \Delta$.

If \mathcal{U} is a uniform structure on X then the pair (X, \mathcal{U}) is called a *uniform space*.

DEFINITION 3.4 A *base* of a uniform structure \mathcal{U} in X is a family $\mathcal{B} \subset \mathcal{U}$ such, that for all $V \in \mathcal{U}$ there exists $W \in \mathcal{B}$ satisfying $W \subset V$.

Each base \mathcal{B} has following properties:

- B1) $(V_1, V_2 \in \mathcal{B}) \Rightarrow (\exists V \in \mathcal{B} [V \subset V_1 \cap V_2])$;
- B2) $\forall V \in \mathcal{B} \exists W \in \mathcal{B} [2W \subset V]$;
- B3) $\bigcap \mathcal{B} = \Delta$.

On the other hand it can be easy proved that if a family \mathcal{B} of neighbourhoods of the diagonal of a set X fulfils conditions (B1)-(B3) then there exists exactly one uniform structure \mathcal{U} on X such that \mathcal{B} is a base of \mathcal{U} .

Every neighbourhood $V \in \mathcal{D}_X$ of the diagonal defines the covering $\mathcal{P}(V) = \{K(x, V)\}_{x \in X}$ of the set X . If \mathcal{U} is a uniform structure in X then every covering \mathcal{O} of X for which there exists $V \in \mathcal{U}$ such that $\mathcal{P}(V)$ is a refinement of \mathcal{O} is said to be a *uniform covering* (with respect to \mathcal{U}).

Each uniform structure on X defines a topology on X . In other words each uniform space (X, \mathcal{U}) defines a topological space (X, Θ) .

THEOREM 3.1 If \mathcal{U} is a uniform structure on X , then a family $\Theta = \{G \subset X : \forall x \in G \exists V \in \mathcal{U} [K(x, V) \subset G]\}$ is a topology in X and (X, Θ) is T_1 -space. A topology Θ is said to be a topology given in X by uniform structure \mathcal{U} and is denoted by $\tau_{\mathcal{U}}$.

For the proof see [1] or [2].

It can be proved that a topology τ on a topological space X is given by some uniform structure on X if and only if X is a Tichonov space (see [2]).

Let ϱ be a pseudometric on a uniform space (X, \mathcal{U}) . If for every $\varepsilon > 0$ there is $V \in \mathcal{U}$ such that if $|x - y| < V$ then $\varrho(x, y) < \varepsilon$, then ϱ is called a *uniform pseudometric (with respect to \mathcal{U})*.

We can defined a uniform structure on three different ways: (i) if we give a base; (ii) if we give a family of uniform coverings or (iii) if we give a family of pseudometrics (see [2]).

Let (M, \mathcal{C}) be a differential space such that $\mathcal{C} = \text{sc}\mathcal{F}_M$ and $(M, \tau_{\mathcal{C}})$ is a Hausdorff space (the last is true iff the family \mathcal{C} separates points in X iff the family \mathcal{F} separates points in X). On the set M the family \mathcal{F} defines the uniform structure $\mathcal{U}_{\mathcal{F}}$ such that the base \mathcal{B} of $\mathcal{U}_{\mathcal{F}}$ is given as follows:

$$\mathcal{B} = \{V(f_1, \dots, f_k, \varepsilon) \subset M \times M; k \in \mathbf{N}; f_1, \dots, f_k \in \mathcal{F}, \varepsilon > 0\},$$

where

$$V(f_1, \dots, f_k, \varepsilon) = \{(x, y) \in M \times M : \forall 1 \leq i \leq k \quad |f_i(x) - f_i(y)| < \varepsilon\}.$$

PROPOSITION 3.1 *The family \mathcal{B} satisfies on M conditions B1 - B3.*

Proof. (B1) Let: $V_1 = V(f_1, \dots, f_k, \varepsilon_1) \in \mathcal{B}$, $V_2 = V(g_1, \dots, g_n, \varepsilon_2) \in \mathcal{B}$ and $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then

$$V := V(f_1, \dots, f_k, g_1, \dots, g_n, \varepsilon) =$$

$$\{(x, y) \in M \times M : \forall 1 \leq i \leq k \quad |f_i(x) - f_i(y)| < \varepsilon \wedge \forall 1 \leq j \leq n \quad |g_j(x) - g_j(y)| < \varepsilon\} \in \mathcal{B}$$

and $V \subset V_1 \cap V_2$.

(B2) Let $V = V(f_1, \dots, f_k, \varepsilon) \in \mathcal{B}$. Then $W := V(f_1, \dots, f_k, \frac{\varepsilon}{2}) \in \mathcal{B}$ and

$$2W =$$

$$\{(x, y) \in M \times M : \exists z \in M \forall 1 \leq i \leq k \quad |f_i(x) - f_i(z)| < \frac{\varepsilon}{2} \wedge |f_i(z) - f_i(y)| < \frac{\varepsilon}{2}\}$$

$$\subset \{(x, y) \in M \times M : \forall 1 \leq i \leq k \quad |f_i(x) - f_i(y)| < \varepsilon\} = V.$$

(B3) Since for any $V \in \mathcal{B}$ there is $\Delta \subset V$ we have

$$\Delta \subset \bigcap \mathcal{B}.$$

On the other hand

$$\bigcap \mathcal{B} \subset \bigcap_{f \in \mathcal{F}, \varepsilon > 0} V(f, \varepsilon) =$$

$$\{(x, y) \in M \times M : \forall f \in \mathcal{F} \quad \forall \varepsilon > 0 \quad |f(x) - f(y)| < \varepsilon\} =$$

$$\{(x, y) \in M \times M : \forall f \in \mathcal{F} \quad [f(x) = f(y)]\} =$$

$$\{(x, x) \in M \times M\} = \Delta. \quad \square$$

The uniform space $(M, \mathcal{U}_{\mathcal{F}})$ is said to be *the uniform space given by the family of function \mathcal{F}* .

If we have two different families \mathcal{F}_1 , and \mathcal{F}_2 of generators of differential space (M, \mathcal{C}) then the uniform structures $\mathcal{U}_{\mathcal{F}_1}$ and $\mathcal{U}_{\mathcal{F}_2}$ can be different too.

EXAMPLE 3.1 Let $M = \mathbf{R}$, $\mathcal{C} = C^\infty(\mathbf{R})$, $\mathcal{F}_1 = \{id_{\mathbf{R}}\}$ and $\mathcal{F}_2 = \{id_{\mathbf{R}}, f\}$, where

$$id_{\mathbf{R}}(x) = x, \quad \text{and} \quad f(x) = x^2, \quad x \in \mathbf{R}.$$

Then does not exists $\varepsilon > 0$ such that $V(id_{\mathbf{R}}, \varepsilon) \subset V(f, 1)$. Hence $V(f, 1) \notin \mathcal{U}_{\mathcal{F}_1}$ and $\mathcal{U}_{\mathcal{F}_1} \neq \mathcal{U}_{\mathcal{F}_2}$. \square

4 Uniform continuous mapping

Let (X, \mathcal{U}) , (Y, \mathcal{V}) , (X, U) , (Y, V) be uniform spaces.

DEFINITION 4.1 A mapping $f : X \rightarrow Y$ is said to be *uniform* with respect to uniform structures \mathcal{U} and \mathcal{V} if

$$\forall V \in \mathcal{V} \exists U \in \mathcal{U} \forall x, x' \in X [|x - x'| < U \Rightarrow |f(x) - f(x')| < V].$$

In other words for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $U \subset (f \times f)^{-1}(V)$. We denote it by

$$f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V}).$$

It is easy to prove that:

- (i) any uniform mapping $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ is continuous with respect to topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$;
- (ii) a superposition of uniform mappings is a uniform mapping.

We can give criteria of the uniformity:

THEOREM 4.1 Let $f : X \rightarrow Y$ and let \mathcal{U} and \mathcal{V} be uniform structures on X and Y respectively. Then the following conditions are equivalent:

- (a) $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$.
- (b) If \mathcal{B} and \mathcal{D} are bases of \mathcal{U} and \mathcal{V} respectively then for each $V \in \mathcal{D}$ there exists $U \in \mathcal{B}$ such that $U \subset (f \times f)^{-1}(V)$.
- (c) For every covering \mathcal{A} of Y uniform with respect to \mathcal{V} , a covering $\{f^{-1}(A)\}_{A \in \mathcal{A}}$ of X is uniform with respect to \mathcal{U} .

(d) For every pseudometric ϱ on Y uniform with respect to \mathcal{V} , a pseudometric σ on X given by the formula

$$\sigma(x, y) = \varrho(f(x), f(y)) \quad x, y \in X$$

is uniform with respect to the uniform structure \mathcal{U} .

For the proof see [2].

A mapping f , that is a uniform with respect to uniform structures \mathcal{U} and \mathcal{V} could be not uniform with respect to uniform structures $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$.

Example 4.1. Let $M = \mathbf{R}$, $C = C^\infty(\mathbf{R})$, $\mathcal{F}_1 = \{id_{\mathbf{R}}\}$, $\mathcal{F}_2 = \{id_{\mathbf{R}}, f\}$, where $f(x) = x^2$, $x \in \mathbf{R}$.

Here f is the uniform mapping with respect to \mathcal{F}_2 , but it is not uniform with respect to \mathcal{F}_1 . In fact, the set $V = \{(x, y) : |f(x) - f(y)| = x^2 - y^2 < \varepsilon\}$ is an element of \mathcal{D} and does not exists $U \in \mathcal{B}$ such that $\mathcal{U} \subset (f \times f)^{-1}(V)$ (see Example 3.1).

DEFINITION 4.2 A bijective mapping $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ is a *uniform homeomorphism* if f^{-1} is a uniform mapping. Then we say that (X, \mathcal{U}) and (Y, \mathcal{V}) are *uniformly homeomorphic*.

By (i) it is obvious that if $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ is a uniform homeomorphism then f is a homeomorphism of the topological spaces $(X, \tau_{\mathcal{U}})$ and $(Y, \tau_{\mathcal{V}})$.

5 Complete uniform spaces and extensions of differential structure

DEFINITION 5.1 Let X be a non empty set, $x \in X$ and $V \in \mathcal{D}_X$ (see Definition 3.1). A set $U \subset X$ is said to be *small of rank V* if $\exists x \in V [U \subset K(x, V)]$ (see Definition 3.2).

DEFINITION 5.2 A nonempty family \mathcal{F} of subsets of a set X is said to be a *filter* on X if:

$$(F1) (F \in \mathcal{F} \wedge F \subset U \subset X) \Rightarrow (U \in \mathcal{F});$$

$$(F2) (F_1, F_2 \in \mathcal{F}) \Rightarrow (F_1 \cap F_2 \in \mathcal{F});$$

$$(F3) \emptyset \notin \mathcal{F}.$$

DEFINITION 5.3 A *filtering base* on X is a nonempty family \mathcal{B} of subsets of X such that

(FB1) $\forall A_1, A_2 \in \mathcal{B} \exists A_3 \in \mathcal{B} [A_3 \subset A_1 \cap A_2]$;

(FB2) $\emptyset \notin \mathcal{B}$.

If \mathcal{B} is a filtering base on X then

$$\mathcal{F} = \{F \subset X : \exists A \in \mathcal{B} [A \subset F]\}$$

is a filter on X . It is called *the filter defined by \mathcal{B}* .

DEFINITION 5.4 Let X be a topological space. We say that a filter \mathcal{F} on X is *convergent to $x \in X$* ($\mathcal{F} \rightarrow x$) if for any neighbourhood U of x there exists $F \in \mathcal{F}$ such that $F \subset U$.

DEFINITION 5.5 Let (X, \mathcal{U}) be a uniform space. A filter \mathcal{F} on X is a *Cauchy filter* if

$$\forall V \in \mathcal{U} \exists F \in \mathcal{F} [F \times F \subset V].$$

DEFINITION 5.6 A uniform space (X, \mathcal{U}) is said to be *complete* if each Cauchy filter on X is convergent in $\tau_{\mathcal{U}}$.

Let (X, \mathcal{U}) be a uniform space, $M \subset X$ and $M \neq \emptyset$. Denote

$$\mathcal{U}_M := \{V \cap M : V \in \mathcal{U}\}.$$

Then it is easy to show that \mathcal{U}_M is a uniform structure on M . We call (M, \mathcal{U}_M) a *uniform subspace of the uniform space (X, \mathcal{U})* .

THEOREM 5.1 *If (X, \mathcal{U}) is a complete uniform space and M is a closed subset of the topological space $(X, \tau_{\mathcal{U}})$ then a uniform space (M, \mathcal{U}_M) is complete. Conversely, If (M, \mathcal{U}_M) is a complete uniform subspace of some (not necessarily complete) uniform space (X, \mathcal{U}) then M is closed in X with respect to $\tau_{\mathcal{U}}$.*

For the proof see [1], [2] or [3].

Any uniform space can be treated as a uniform subspace of some complete uniform space. We have the following

THEOREM 5.2 *For each uniform space (X, \mathcal{U}) :*

- (i) *there exists a complete uniform space $(\tilde{X}, \tilde{\mathcal{U}})$ and a set $A \subset \tilde{X}$ dense in \tilde{X} (with respect to the topology $\tau_{\tilde{\mathcal{U}}}$) such that (X, \mathcal{U}) is uniformly homeomorphic to $(A, \tilde{\mathcal{U}}_A)$;*
- (ii) *if the complete uniform spaces $(\tilde{X}_1, \tilde{\mathcal{U}}_1)$ and $(\tilde{X}_2, \tilde{\mathcal{U}}_2)$ satisfies condition of the point (i) then they are uniformly homeomorphic.*

For the details of the proof see [1] or [3]. Here we only want to describe the construction of $(\tilde{X}, \tilde{\mathcal{U}})$.

Let \tilde{X} be the set of all minimal (with respect to the order defined by inclusion) Cauchy filters in X . For every symmetric set $V \in \mathcal{U}$ we denote by \tilde{V} the set of all pairs $(\mathcal{F}_1, \mathcal{F}_2)$ of minimal Cauchy's filters, which have a common element being a small set in rank V . We define a family $\tilde{\mathcal{U}}$ of subsets of set $\tilde{X} \times \tilde{X}$ as the smallest uniform structure on X containing all sets from the family $\{\tilde{V} : V \in \mathcal{U}\}$.

Let us consider two different uniform structures at the same differential space $(\mathbf{R}, \mathcal{C}^\infty)$: $\mathcal{U}_{\mathcal{F}}$ and $\mathcal{U}_{\mathcal{G}}$, where $\mathcal{F} = \{id_{\mathbf{R}}\}$, $\mathcal{G} = \{arctgx\}$. Then $(\mathbf{R}, \mathcal{U}_{\mathcal{F}})$ is the complete space $(\tilde{\mathbf{R}} = \mathbf{R})$ whereas $(\mathbf{R}, \mathcal{U}_{\mathcal{G}})$. In this case we can identify $\tilde{\mathbf{R}}$ with the interval $[-\frac{\pi}{2}; \frac{\pi}{2}]$.

Let N be a set, $M \subseteq N$, $M \neq \emptyset$, \mathcal{C} be a differential structure on M .

DEFINITION 5.7. The differential structure \mathcal{D} on N is an *extension* of the differential structure \mathcal{C} from the set M to the set N if $\mathcal{C} = \mathcal{D}_M$ (if we get the structure \mathcal{C} by localization of the structure \mathcal{D} to M).

For the sets N, M and the differential structure \mathcal{C} on M we can construct many different extensions of the structure M to N .

EXAMPLE 5.1. If for each function $f \in \mathcal{C}$ we assign the function $f_0 \in \mathbf{R}^N$ such that $f_{0|M} = f$ and $f_{0|N \setminus M} \equiv 0$. Then the differential structure generated on N by the family of functions $\{f_0\}_{f \in \mathcal{C}}$ is the extension of \mathcal{C} from M to N . Similarly, if for each function $f \in \mathcal{C}$ we assign the family of the functions $\mathcal{F}_f := \{g \in \mathbf{R}^N : g|_M = f\}$, then the differential structure on N generated the family of the functions $\mathcal{F} := \bigcup_{f \in \mathcal{C}} \mathcal{F}_f$ is the extension of \mathcal{C} from M to N . If the set $N \setminus M$ contains at least two elements, then the differential structures generated by the families $\{f_0\}_{f \in \mathcal{C}}$ and \mathcal{F} are different.

DEFINITION 5.8. If τ is a topology on the set N , then the extension \mathcal{D} of the differential structure \mathcal{C} from M to N is *continuous with respect to τ* if each function $f \in \mathcal{D}$ is continuous in the topology τ ($\tau_{\mathcal{D}} \subset \tau$).

If on the set N there exists continuous (with respect to τ) extension of the differential structure \mathcal{C} from the set $M \subset N$, then the structure \mathcal{C} is said to be *extendable from the set M to the topological space (N, τ)* .

EXAMPLE 5.2. The differential structure $C^\infty(\mathbf{R})_{\mathbf{Q}}$ is extendable from the set of rationales to the set of reals. The continuous extensions are e.g. $C^\infty(\mathbf{R})$ and the structure \mathcal{D} generated on \mathbf{R} by the family of functions $C^\infty(\mathbf{R}) \cup \{f\}$, where $f : \mathbf{R} \rightarrow \mathbf{R}$, $f(x) := |x - \sqrt{2}|$, $x \in \mathbf{R}$.

It is not difficult to show that if \mathcal{F} is a family of generators of a differential structure \mathcal{C} on a set M then the completion \tilde{M} of M with respect to the uniform structure $\mathcal{U}_{\mathcal{F}}$ can be identify with the closure of the range $\phi_{\mathcal{F}}(M)$ of the generator embedding $\phi_{\mathcal{F}}$ in the Cartesian product $\mathbf{R}^{\mathcal{F}}$. In this case the differential structure $C^\infty(\mathbf{R}^{\mathcal{F}})_{\phi_{\mathcal{F}}(M)}$ is a natural continuous extension of \mathcal{C} from M to \tilde{M} .

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